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Letters and comments

On a paradox concerning the temperature distribution of an ideal gas in a gravitational field

Abstract. A paradox concerning the temperature distribution of an ideal gas in a gravitational field is analysed within the microcanonical ensemble framework. We propose a resolution of this paradox when the cases of a finite system and an infinite system, i.e. a system in the thermodynamic limit, are considered.

Resumen. Se analiza, dentro de la colectividad microcanónica, una paradoja referente a la distribución de temperatura de un gas ideal en un campo gravitatorio. Se propone una resolución de dicha paradoja según que se considere un sistema finito o infinito, es decir, en el límite termodinámico.

In a recent paper (Román *et al* 1995) we have derived, by using the microcanonical (EVN) ensemble, the single-particle distribution density functions for an ideal gas in a gravitational field. More recently, a collegue drew our attention on a very interesting paradox concerning the temperature distribution of an ideal gas in the presence of a gravitational field presented by Coombes and Laue some years ago (Coombes and Laue 1985). The problem proposed and analysed by these authors is the following: If a vertical column of an *adiabatically enclosed* ideal gas is in thermal equilibrium, is the temperature the same throughout the column or is there a temperature gradient along the direction of the gravitational field? According to Coombes and Laue, there are two conflicting answers to the above question:

(1) The temperature is the same throughout because the system is in equilibrium.

(2) The temperature decreases with the height because of the following two reasons.

  (a) Energy conservation implies that every molecule loses kinetic energy as it travels upward, so that the average kinetic energy of all molecules decreases with height.

  (b) Temperature is proportional to the average molecular kinetic energy.

Coombes and Laue concluded that answer (1) is the correct one and answer (2) is wrong. They reached this conclusion after finding that statement (2a) is wrong, i.e., the average kinetic energy of all molecules does not decrease with the height even though the kinetic energy of each individual molecule does decrease with height. These authors give at first a qualitative explanation of this fact by noting that since both the kinetic energy of the molecules and the number density of molecules decrease with height, the average molecular kinetic energy does not necessarily decrease with height. Indeed, if one considers a volume element at a given height z, the *average* molecular kinetic energy $\langle K\rangle$ corresponding to this volume element is:

$$\langle K(z)\rangle = \frac{\sum_z K}{N(z)} \tag{1}$$

where $N(z)$ is the number of molecules in the volume element and the summation is over all possible values for the kinetic energy of these molecules. From (1), it is evident that $\langle K(z)\rangle$ can remain constant (i.e. z-independent) even though both $\sum_z K$ and $N(z)$ decrease with height. Obviously, the z-dependence of $\sum_z K$ and $N(z)$ must be such that this variable cancels in the RHS of (1).

A second, more quantitative, explanation is given by considering the the functional form of the height and velocity distribution density function, $g(z, v)$, for an ideal gas in an uniform gravitational field. The Coombes and Laue procedure is then based in two crucial assumptions:

(i) Variables z v are statistically independent. This means that the distribution $g(z, v)$ can be factorized in the form:

$$g(z, v) = n(z)\omega(v) \tag{2}$$

where $n(z)$ is the height distribution density function and $\omega(v)$ is the velocity distribution density function.

(ii) Distribution $\omega(v)$ has the Maxwell–Boltzmann form. In particular, for a f-dimensional ideal gas ($f = 1, 2$ or 3) one has:

$$\omega(v) = \frac{m^{f/2}}{\Gamma(\frac{f}{2})2^{(f-2)/2}} \times \left(\frac{1}{k_B T}\right)^{f/2} v^{f-1} \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{3}$$

(We note that in (2) $n(z)$ has the well known *barometric* form, $n(z) = (mg/k_B T)\exp(-mgz/k_B T)$, but its use will not be necessary in the following.)

By using (2) and (3), if one considers the volume element between z and $z + \mathrm{d}z$, from (1) one obtains,

$$\langle K(z)\rangle = \frac{1}{2} m \frac{\left[\int_0^\infty v^2 g(z, v)\mathrm{d}v\right]\mathrm{d}z}{n(z)\mathrm{d}z} = \frac{f k_B T}{2} \tag{4}$$

which corroborates the assumption that the temperature is proportional to the average molecular kinetic energy.

At this point, we think that an essential question arises: are the above assumptions (i) and (ii) right? The answer is: *yes, if one is working in the*

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canonical ensemble. Indeed, the single-particle velocity distribution has the Maxwell–Boltzmann form and factorization (2) is fulfilled in the *canonical* ensemble. We remember that the canonical emsemble describes a system in equilibrium with a thermal bath (or heat reservoir). But the question formulated by Coombes and Laue concerns an *adiabatically enclosed* system, i.e. a system with N molecules in a fixed volume V having constant total energy E. As is well known, such a system is described by the *microcanonical* ensemble. Then, a more accurate re-statement of the above question is: do assumptions (i) and (ii) hold in the microcanonical ensemble?

The aim of this Letter is to analyse the above question in the light of our recent results reported in [1]. In particular, the following microcanonical single-particle distributions for a f-dimensional ideal gas in a gravitational field have been derived (see (12), (14) and (15) in Román *et al* (1995)):

$$g_M(z,v) = \frac{\Gamma(\frac{fN}{2}+N)}{\Gamma(\frac{f}{2})\Gamma(\frac{fN}{2}+N-\frac{f+2}{2})} \times \frac{m^{(f+2)/2} g v^{f-1}}{2^{(f-2)/2}E^{(f+2)/2}} \times \left(1-\frac{mv^2}{2E}-\frac{mgz}{E}\right)^{(\frac{f}{2}+1)N-(\frac{f}{2}+2)} \quad (5)$$

$$n_M(z) = \left(\frac{fN}{2}+N-1\right)\frac{mg}{E}\left(1-\frac{mgz}{E}\right)^{(\frac{f}{2}+1)N-2} \quad (6)$$

$$\omega_M(v) = \frac{\Gamma(\frac{fN}{2}+N)}{\Gamma(\frac{f}{2})\Gamma(\frac{fN}{2}+N-\frac{f}{2})} \times \frac{m^{f/2}v^{f-1}}{2^{(f-2)/2}E^{f/2}}\left(1-\frac{mv^2}{2E}\right)^{(\frac{f}{2}+1)(N-1)} \quad (7)$$

where the subscript M denotes *microcanonical*. A key aspect of the microcanonical distributions (5)–(7), relevant to the present discussion, is that they have the following two properties:

- Variables z and v are not statistically independent. (One easily checks that: $g_M(z,v) \neq n_M(z)\omega_M(v)$)
- Distribution $\omega_M(v)$ has not the Maxwell–Boltzmann form.

The above properties imply, therefore, that assumptions (i) and (ii) are not applicable in the microcanonical analysis of a *finite* system. Then, the following step is to derive the average molecular kinetic energy (1) by using the microcanonical distributions. By using (5) and (6), one gets,

$$\langle K(z)\rangle = \frac{1}{2}m\frac{\left[\int_0^{\sqrt{\frac{2(E-mgz)}{m}}} v^2 g_M(z,v)\,\mathrm{d}v\right]\mathrm{d}z}{n_M(z)\,\mathrm{d}z} = \frac{fE}{fN+2N-2}\left(1-\frac{mgz}{E}\right) \quad (8)$$

i.e., *for a finite adiabatically enclosed ideal gas in a gravitational field the average molecular kinetic energy decreases with height*.

Then the question now is: under what conditions the Coombes and Laue explanation of the formulated paradox can be applied? The answer is clear: *in the thermodynamic limit*, i.e., when

$$N \to \infty\,,\ E \to \infty\,,\ \frac{E}{N} = finite \neq 0 \quad (9)$$

Indeed, we have shown (see (41) in Román *et al* (1995)) that, in the microcanonical ensemble, the temperature of the system under consideration is given by

$$\frac{1}{k_B T} = \left(\frac{f}{2}+1\right)\frac{N}{E} \quad (10)$$

which remains finite in the thermodynamic limit. Then, by substituting (10) into (8) and taking the thermodynamic limit one easily obtains the result (4), i.e. the temperature is proportional to the average molecular kinetic energy at any height z in the thermodynamic limit. This is a particular consequence of the fact that in the thermodynamic limit the *microcanonical* distributions must approach the *canonical* forms, since the infinite system acts as a thermal bath for an individual molecule.

In conclusion, in our opinion a full explanation about why answer (2) to the paradox formulated by Coombes and Laue is wrong must discern between the cases of a *finite* system and an *infinite* system. In the former case, statement (2) is wrong because the assumption in statement (2b) is wrong. In the latter case, statement (2) is wrong because the conclusion in statement (2a) is wrong (as it has been established by Coombes and Laue).

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