

A paradox concerning the temperature distribution of a gas in a gravitational field

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In this Note we will present an apparent paradox that we have found useful in teaching some properties of distribution functions. The paradox is easy to formulate. However, its resolution is not so easy to find. Still, students find the solution entirely satisfactory when it is explained carefully. Since most students find a paradox challenging and interesting, paradoxes are good for the learning process. The paradox consists in the possibility of having two apparently meaningful but conflicting answers to the following question.

Question: If a vertical column of air which is adiabatically enclosed is in thermal equilibrium, is the temperature the same throughout the column or is there a temperature gradient along the direction of the gravitational field?—Let us assume an idealized model in which there are no interactions between air molecules and no collisions once the equilibrium distribution has been established. The resolution of the paradox below is unrelated to this assumption. Energy conservation implies that the kinetic energy (KE) of a molecule decreases with its height z above the Earth's surface according to the formula

$$KE = (KE)_0 - mgz, \quad (1)$$

where m is the mass of the molecule and g is the acceleration due to gravity. Now, the two conflicting answers to the question above are as follows.

(1) There is no temperature gradient because a system which is in thermal equilibrium has the same temperature throughout.

(2) The temperature decreases as one goes up the column for the following reasons. (a) Since Eq. (1) implies that every molecule loses kinetic energy as it moves upward, the average kinetic energy of all molecules decreases with height. (b) Temperature is proportional to the average molecular kinetic energy. Combining (a) and (b) establishes statement (2) above.

Which of the two statements (1) and (2) is correct? Where lies the mistake? Note that the argument is short and that it is hard to find anything wrong with any of the statements. In fact, the second arguments seem so convincing that one begins to doubt the validity of statement (1) in the presence of a gravitational field. We recommend that students be made to wait from one period to the next before the solution is given. We also recommend that the reader of this Note spend some time carefully thinking about the problem before continuing.

The answer is that statement (1) is right and statement (2) wrong. Statement (2) is wrong because the conclusion in statement (2a) is wrong. The seemingly paradoxical truth is that the *average* kinetic energy of all molecules does *not* decrease with height even though the kinetic energy of each individual molecule does decrease with height.

In explaining this effect let us note first that the number of molecules per unit volume must decrease with height; slower molecules fall back down after a distance depending on their initial upward velocity component. In going from the premise in statement (2a) to the conclusion in statement

(2a), one tacitly assumes that the same molecules are used in comparing averages at lower and higher elevation. Clearly, if we were considering a fixed number N of molecules, the definition of average kinetic energy \overline{KE} ,

$$\overline{KE} = (\sum KE)/N, \quad (2)$$

would imply for smaller $\sum KE$ a smaller value for \overline{KE} . However, we now realize that not all N molecules present in a volume element at lower elevation reach a corresponding volume element at higher elevation; thus \overline{KE} need not necessarily decrease if both $\sum KE$ and N decrease. Therefore the conclusion in statement (2a) was reached too hastily. Nevertheless, all molecules will have a smaller energy at the higher elevation than they had at the lower elevation, and one still has the feeling that the conclusion in statement (2a) might be valid. It is not! We will now show why.

Let us restrict the discussion to a one-dimensional model in which molecules can only move in the vertical direction. To generalize the discussion to three dimensions requires little additional effort.

In answering the question posed above, we will also derive some other related facts. Temperature is a parameter characterizing the molecular velocity distribution at thermal equilibrium. The column of air is supposed to be in thermal equilibrium at every height. Thus we could assume that the velocity distribution function $f(z, v)$ which is normalized so that

$$f(z, v) dz dv = \text{number of molecules in interval } (z, z + dz) \text{ with velocities in interval } (v, v + dv) \quad (3)$$

is proportional to the Maxwell velocity distribution at every z . However, we shall only assume a Maxwell velocity distribution at one height z_1 ,

$$f(z_1, v) = C \exp\left(-\frac{mv^2}{2kT}\right). \quad (4)$$

T is the temperature at z_1 and C is proportional to the density at z_1 . We now show that assumption (4) implies a Maxwellian velocity distribution for other heights as well with the same temperature even though no collisions are occurring.

It is shown in the Appendix that the value of the distribution function f at height z_2 and velocity v_2 is equal to its value at height z_1 and velocity v_1 if v_2 , z_2 , v_1 , and z_1 are related by Eq. (1), i.e.,

$$f(z_2, v_2) = f(z_1, v_1) \quad (5)$$

if

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v_2^2 - mg(z_1 - z_2). \quad (6)$$

By placing (6) into (4) and then using this in (5), one obtains that

$$f(z_2, v_2) = \left[C \exp\left(\frac{mgz_1}{kT}\right) \right] \exp\left(-\frac{mgz_2}{kT}\right) \exp\left(-\frac{mv_2^2}{2kT}\right). \quad (7)$$

Since, for given z_1 , the factor in curly brackets in expression (7) is a constant and since z_2 and v_2 are independent variables, expression (7) already provides us with the desired result. In addition, formula (7) after integration over v_2 gives us the concentration as a function of z_2 . Since the concentration is proportional to the pressure of the gas, a by-product of our discussion is a derivation of the barometric pressure formula.

In trying to describe these calculations verbally, one might say the following. The kinetic energy of all molecules decreases by a constant amount as they rise through a constant height. Since the Maxwell distribution depends exponentially on the kinetic energy, a shift in kinetic energy by a constant amount translates into an overall scale factor with no change in distribution of velocities. For example, if v'_1 and v'_2 are speeds of two different molecules at height z_1 and if the probability of finding v'_1 is twice that of finding v'_2 , then the probability of finding v'_2 will still be twice the probability of finding v'_1 . Since the probability distributions are equal, the temperatures are equal. The net effect of the gravitational field is to modify the concentration of particles in every velocity range by a velocity-independent scale factor whose value decreases with height in a manner expressed quantitatively by the barometric pressure formula. The phenomenon discussed can also be described as a kind of "self-similarity" property of the Maxwell distribution.

The problem presented here and its solution describe a mechanism which explains the mathematical form of a certain distribution function. The distribution function $f(z, v)$ can also be obtained by solving the Boltzmann equation with an external force field present.¹ However, few undergraduate students would appreciate the physics underlying such a mathematical treatment. The paradox presented here is very interesting to most students so that they are willing to spend the time necessary to clear up the details. The process becomes a good learning experience and provides a good background for more formal treatments later on.

In generalizing the discussion to three dimensions one uses Cartesian coordinates. The horizontal coordinates x , y , v_x , v_y can simply be "tacked on" to the treatment above because the Maxwell distribution and the volume element in (\mathbf{r}, \mathbf{v}) space both factorize. We recommend as an exercise for the students after they have been exposed to the explanation given above to write out all steps for the three-dimensional case.

APPENDIX

At any given time, there are $f(z_1, v_1) dz_1 dv_1$ molecules with (z, v) values in the rectangle

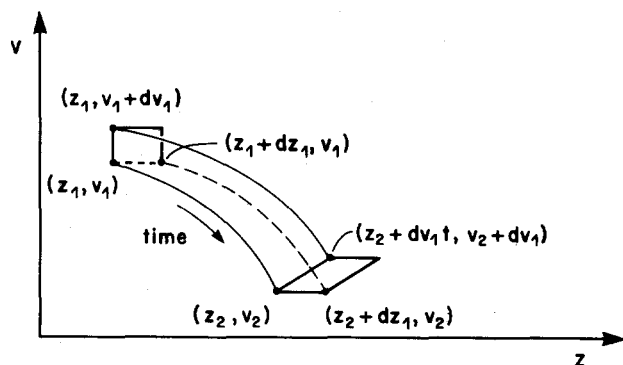


Fig. 1. Time evolution of a volume element in (z, v) space when a uniform gravitational field is present.

$$z_1 < z < z_1 + dz_1, \quad (8)$$

$$v_1 < v < v_1 + dv_1.$$

Let us consider the particles that are in rectangle (8) at time $t = 0$ and let us ask what is the "volume element" in (z, v) space occupied by the same particles at a later time t . Under the influence of the constant gravitational field, a particle with coordinates $[z(0), v(0)]$ at time $t = 0$ acquires in time t the coordinates

$$z(t) = z(0) + v(0)t - (g/2)t^2, \quad (9)$$

$$v(t) = v(0) - gt.$$

Figure 1 illustrates the time development of the (z, v) coordinates of particles initially in rectangle (8). The figure shows that the rectangle at a later time t becomes a parallelogram. The coordinates of three of the corners of the rectangle and parallelogram are indicated in Fig. 1. The (z, v) values of corresponding corner points are related by Eqs. (9). We recommend as an exercise for the students to check that rectangle (8) has indeed become a parallelogram at time t by calculating the coordinates of the fourth corner of the parallelogram in terms of the coordinates at $t = 0$. Note that the areas of the initial rectangle and final parallelogram are equal because the rectangle and the parallelogram both have width dz_1 and height dv_1 . Denoting the area $dz_1 dv_1$ of this common "volume element" by $d\tau$, particle number conservation gives us

$$f(z_1, v_1) d\tau = f(z_2, v_2) d\tau$$

or

$$f(z_1, v_1) = f(z_2, v_2). \quad (10)$$

Formula (10) is the same as formula (5) in the text.

¹S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University, Cambridge, 1970), 3rd ed., Sec. 4.14.